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Journal of Combinatorial Theory, Series B 95 (2005) 152–167
Journal of Combinatorial Theory Series B
www.elsevier.com/locate/jctb

Improvements of the theorem of Duchet and Meyniel on Hadwiger's conjecture

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Received 31 December 2003
Available online 31 May 2005

Abstract

Since $\chi(G)\cdot\alpha(G)\geqslant n(G)$, Hadwiger's conjecture implies that any graph G has the complete graph $K_{\lceil n/\alpha\rceil}$ as a minor, where $n=n(G)$ is the number of vertices of G and $\alpha=\alpha(G)$ is the maximum number of independent vertices in G. Duchet and Meyniel [Ann. Discrete Math. 13 (1982) 71–74] proved that any G has $K_{\lceil n/(2\alpha-1)\rceil}$ as a minor. For $\alpha(G)=2$ G has $K_{\lceil n/3\rceil}$ as a minor. Paul Seymour asked if it is possible to obtain a larger constant than $\frac{1}{3}$ for this case. To our knowledge this has not yet been achieved. Our main goal here is to show that the constant $1/(2\alpha-1)$ of Duchet and Meyniel can be improved to a larger constant, depending on α, for all $\alpha\geqslant 3$. Our method does not work for $\alpha=2$ and we only present some observations on this case.

Keywords: Hadwiger's conjecture; Theorem of Duchet and Meyniel; Independence number; Connected matching

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[1] Research of Ken-ichi Kawarabayashi was partly supported by the Japan Society for the Promotion of Science for Young Scientists. The research was carried out in May 2002 and October 2003 at the University of Southern Denmark, supported by the Danish Natural Science Research Council.

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doi:10.1016/j.jctb.2005.04.001

1. Introduction

Let $\chi(G)$ denote the chromatic number of a graph G. In a $\chi(G)$-colouring of G each colour-class has size at most $\alpha(G)$, the size of a maximum independent set of G. Hence $\chi(G) \geqslant |V(G)|/\alpha(G)$.

Hadwiger's conjecture states that any graph G has the complete graph $K_{\chi(G)}$ as a minor. Hence, by the above inequality, Hadwiger's conjecture implies

Conjecture 1.1. *Any non-empty graph G on n vertices has $K_{\lceil n/\alpha(G)\rceil}$ as a minor.*

This conjecture seems weaker than Hadwiger's conjecture, however for $\alpha(G) = 2$ the two conjectures are equivalent [7]. Conjecture 1.1 was explicitly stated and put into context by Woodall [9].

In 1982 Duchet and Meyniel [2] proved

Theorem of Duchet and Meyniel. *Any non-empty graph G on n vertices has $K_{\lceil n/(2\alpha(G)-1)\rceil}$ as a minor.*

The proof is by induction on *n* and based on the fact that any connected non-empty graph G has a subset $T \subseteq V(G)$ such that the subgraph $G[T]$ of G induced by T is connected, $|T| \leqslant 2\alpha(G) - 1$, and such that T is *dominating*, i.e. any vertex of $G - T$ has a neighbour in T. The set T is built up stepwise from one vertex, in each step adding two new joined vertices, one from the neighbourhood of T and one from the non-neighbourhood of T. Since the independence number of $G[T]$ is increased by one in each step, the desired T is obtained after at most $\alpha(G) - 1$ steps.

No essential improvement of the theorem of Duchet and Meyniel has been obtained, although Maffray and Meyniel [6] obtained related results. The aim of the present paper is to prove (with $\omega(G)$ denoting the number of vertices in a maximum complete subgraph of G).

Theorem 1.2. *Any non-empty graph G on n vertices with $\alpha(G) \geqslant 2$ has $K_{\lceil (n+w(G))/(2\alpha(G)-1)\rceil}$ as a minor.*

Theorem 1.3. *Any non-empty graph G on n vertices with $\alpha(G) \geqslant 3$ has $K_{\lceil n(1+c)/(2\alpha(G)-1)\rceil}$ as a minor for some $c > 0$, c depending on $\alpha(G)$.*

Theorem 1.2 for $\alpha(G) = 2$ was first obtained by Plummer et al. [7]. Since $\omega(G)$ for $\alpha(G) = 2$ may be of order of magnitude as small as $\sqrt{n \log n}$ [5], and for $\alpha(G) \geqslant 3$ of order of magnitude as small as $n^{2/\alpha(G)} \cdot \log n$ [3], this does not improve the constant in Duchet and Meyniel's theorem.

Paul Seymour asked a few years ago for an improvement of the constant $\frac{1}{3}$ in the theorem of Duchet and Meyniel in the case $\alpha(G) = 2$. As far as we know no such improvement has been achieved so far. Theorem 1.3 shows however that for $\alpha(G) \geqslant 3$ such an improvement is possible. Our main goal here is to point out the existence of the positive c in

Theorem 1.3. The value our proof gives is

$$c(\alpha) = 1/(4\alpha - 3)$$

and hence we may state Theorem 1.3 in the following form:

Theorem 1.4. *Any non-empty graph G on n vertices with* $\alpha(G) \geqslant 3$ *has* $K_{\lceil n(4\alpha(G)-2)/((4\alpha(G)-3)(2\alpha(G)-1))\rceil}$ *as a minor.*

For $\alpha(G) = 3$ we thus obtain $K_{\lceil 2n/9\rceil}$ as a minor, compared to $K_{\lceil n/5\rceil}$ in the theorem of Duchet and Meyniel.

For $\alpha = 2$ our method does not work. For that case, to improve the constant $\frac{1}{3}$, it is necessary and sufficient to find a large connected matching (a matching is *connected* if any two matching edges are joined by at least one edge). This was noted perhaps first by Thomassé [8]. More explicitly we shall prove:

Theorem 1.5. *If G is a graph on n vertices with* $\alpha(G) \leqslant 2$ *containing a connected matching of size* $\geqslant kn > 0$, *then G has* $K_{\lceil (n/3)(1+k/3)\rceil}$ *as a minor.*

Conversely, if G is a graph on n vertices with $\alpha(G) \leqslant 2$ *having* $K_{\lceil cn\rceil}$ *as a minor for* $c > \frac{1}{3}$, *then G contains a connected matching of size at least* $(3c-1)n/4 - \frac{1}{2}$.

Füredi et al. [4] proved that every graph on $4t - 1$ vertices and $\alpha(G) = 2$ contains a connected matching of size t for $t \leqslant 17$. For general t they proved the same conclusion with $4t - 1$ replaced by a function of order $t^{3/2}$. Cameron [1] obtained complexity results and algorithms for connected matchings.

For the case $\alpha = 2$ we offer only some minor observations on connected matchings, among them

Theorem 1.6. *Let G be a graph on n vertices with* $\alpha(G) \leqslant 2$. *If G is* $(n-4)$*-connected then G has* $K_{\lceil n/2\rceil}$ *as a minor.*

2. Proof of Theorem 1.2

The proof is by induction on $n = |V(G)|$. For small values of n (say $n = 2, 3, 4$) the theorem is true by inspection. For $\alpha(G) = n$ the theorem is obviously true also.

Let $\omega = \omega(G) \geqslant 2$ and $\alpha = \alpha(G) \geqslant 2$. Let K' denote any complete ω'-graph in G, where $1 \leqslant \omega' \leqslant \omega$, and let $H = G - K'$.

Case 1: $\alpha(H) < \alpha$: This assumption and $\alpha(G) = \alpha$ imply that $\alpha(H) = \alpha - 1$. By the result of Duchet and Meyniel H has the complete $\left\lceil \frac{n-\omega'}{2\alpha-3} \right\rceil$-graph as a minor. Also G has K' as a minor. If $\frac{n-\omega'}{2\alpha-3} \geqslant \frac{n+\omega}{2\alpha-1}$ or if $\omega' \geqslant \frac{n+\omega}{2\alpha-1}$, then G has the complete $\left\lceil \frac{n+\omega}{2\alpha-1} \right\rceil$-graph as a minor, and Theorem 1.2 holds in this case.

Hence we may assume that $(n-\omega')(2\alpha-1) < (n+\omega)(2\alpha-3)$ and $(2\alpha-1)\omega' < n+\omega$.

By adding these two inequalities we get

$$n(2\alpha - 1) < (n + \omega)(2\alpha - 2)$$

or

$$n < \omega(2\alpha - 2)$$

or

$$n + \omega < \omega(2\alpha - 1).$$

Thus $\omega > \frac{n+\omega}{2\alpha-1}$ implying that G contains a complete $\left\lceil \frac{n+\omega}{2\alpha-1} \right\rceil$-graph.

This finishes the proof in Case 1.

For the rest of the proof we let $\omega' = \omega$, i.e. K' is a maximum complete subgraph K.

Case 2: $\alpha(H) = \alpha$ *and* H *is disconnected*: Let H be divided into two non-empty parts H_1 and H_2 with no edges between them. Let $\alpha(H_1) = \alpha_1$ and $\alpha(H_2) = \alpha_2$. Then $\alpha = \alpha(H) = \alpha_1 + \alpha_2$.

For a vertex $x \in V(K)$ consider $G[x \cup V(H_1)]$ and $G[x \cup V(H_2)]$. If these two graphs have independent sets of sizes $\alpha_1 + 1$ and $\alpha_2 + 1$, respectively, then $\alpha(G) \geqq \alpha_1 + \alpha_2 + 1 > \alpha$, which is a contradiction.

If $\alpha(G[x \cup V(H_1)]) = \alpha_1$ we let x be of type 1, otherwise it is of type 2 and satisfies $\alpha(G[x \cup V(H_2)]) = \alpha_2$. Let the vertices of type 1 and 2 be the sets T_1 and T_2, respectively. Then $\alpha(G[T_i \cup V(H_i)]) = \alpha_i$ for $i = 1, 2$.

If $\alpha_1 \geqslant 2$ and $\alpha_2 \geqslant 2$ we use induction on the two graphs $G_i = G[T_i \cup V(H_i)]$, $i = 1, 2$. We also have $\omega(G_i) \geqslant |T_i|$. We are through unless

$$\frac{|V(G_i)| + |T_i|}{2\alpha_i - 1} < \frac{n + \omega}{2\alpha - 1} \quad \text{for } i = 1, 2.$$

This implies

$$(2\alpha - 1)(|V(G_i)| + |T_i|) < (n + \omega)(2\alpha_i - 1) \quad \text{for } i = 1, 2.$$

By adding the two inequalities we get

$$(2\alpha - 1)(n + \omega) < (n + \omega)(2\alpha - 2).$$

This is a contradiction.

On the other hand, if, say, $\alpha_1 = 1$, then $G_1 = G[T_1 \cup V(H_1)]$ is a complete graph. If we now use G_1 as K', then $\alpha(G - K') = \alpha_2 = \alpha - 1$, and we are back in Case 1.

Thus Theorem 1.2 has been proved also in Case 2.

Case 3: $\alpha(H) = \alpha$ *and* H *is connected*: Let x be any vertex of H. Starting from x we can, by the argument of Duchet and Meyniel, obtain a set T in $V(H)$, such that

(i) $x \in T \subseteq V(H)$;
(ii) $|T| = 2\alpha' - 1$;
(iii) $\alpha(G[T]) = \alpha'$;
(iv) T is dominating in H;
(v) $G[T]$ is connected.

Suppose first that T is dominating in G. Then contract T into one vertex and consider $G-T$. Assume first that $\alpha(G-T)\geqslant 2$. By induction G has a complete k-graph as a minor, with

$$\begin{aligned}
k &\geqslant \left\lceil \frac{n-(2\alpha'-1)+\omega}{2\alpha(G-T)-1} \right\rceil + 1\\
&\geqslant \left\lceil \frac{n+\omega-(2\alpha'-1)+(2\alpha-1)}{2\alpha-1} \right\rceil\\
&\geqslant \left\lceil \frac{n+\omega}{2\alpha-1} \right\rceil .
\end{aligned}$$

Suppose therefore that T is dominating in G, but that $\alpha(G-T)=1$. Then $G-T$ is complete, hence $G-T=K$ and $T=V(H)$, so that $|T|=2\alpha(H)-1=2\alpha-1$. Contracting T into one vertex a $K_{\omega+1}$ is obtained. Moreover

$$\begin{aligned}
\frac{n+\omega}{2\alpha-1} &= \frac{\omega+2\alpha-1+\omega}{2\alpha-1}\\
&= \frac{2}{2\alpha-1}\omega+1\\
&< \omega+1
\end{aligned}$$

hence again G has a complete $\left\lceil \frac{n+\omega}{2\alpha-1} \right\rceil$-graph as minor.

So we may now assume that T is not dominating in G, hence $\alpha'<\alpha$. Since $\alpha(H)=\alpha(G)=\alpha$ there is a vertex $y\in K$ joined to a vertex x of H. We may assume that T is built starting from this vertex x. Then $T\cup\{y\}$ dominates G. Contract $T\cup\{y\}$ into one vertex and use induction, this time on $G-T-y$. First assume that $\alpha(G-T-y)\geqslant 2$. Then G has a complete k-graph as a minor, with

$$\begin{aligned}
k &\geqslant \left\lceil \frac{n-(2\alpha'-1)-1+\omega-1}{2\alpha(G-T-y)-1} \right\rceil + 1\\
&\geqslant \left\lceil \frac{n-(2\alpha'-1)+(2\alpha-1)+\omega-2}{2\alpha-1} \right\rceil\\
&= \left\lceil \frac{n+\omega+2(\alpha-\alpha'-1)}{2\alpha-1} \right\rceil\\
&\geqslant \left\lceil \frac{n+\omega}{2\alpha-1} \right\rceil ,
\end{aligned}$$

where we use that $\alpha'<\alpha$.

If $\alpha(G-T-y)=1$ then $G-T-y$ is complete, hence of size $\leqslant\omega$, that is $|V(H)-T|\leqslant 1$. Since $\alpha(H)=\alpha>\alpha'=\alpha(G[T])$, in fact $|V(H)-T|=1$ and $V(H)-T$ consists of one vertex z joined to all vertices in $K-y$. Moreover $\alpha'=\alpha-1$. Contracting $T\cup\{y\}$ into one vertex a $K_{\omega+1}$ is obtained. Moreover

$$\begin{aligned}
\frac{n+\omega}{2\alpha-1} &= \frac{\omega+2\alpha'-1+1+\omega}{2\alpha-1}\\
&= \frac{2}{2\alpha-1}\omega+\frac{2\alpha'}{2\alpha-1}
\end{aligned}$$

$$= \frac{2}{2\alpha - 1}\omega + \frac{2\alpha - 2}{2\alpha - 1}$$
$$< \omega + 1.$$

This finishes Case 3, and hence finally proves Theorem 1.2. $\square$

3. Proof of Theorems 1.3 and 1.4

We shall prove Theorem 1.4 which implies Theorem 1.3. Specifically, we prove that any non-empty graph G on n vertices with $\alpha = \alpha(G) \geqslant 3$ has $K_{\lceil n(1+c)/(2\alpha-1)\rceil}$ as a minor, where $c = \frac{1}{4\alpha-3} \leqslant \frac{1}{9}$.

The proof is by induction on n. For $n = \alpha$ we have $n(1 + c)/(2\alpha - 1) < 1$, hence the theorem is trivially true.

For $\alpha < n \leqslant 4\alpha - 3$ we have $\omega(G) \geqslant 2$ and $n(1+c)/(2\alpha-1) \leqslant 2$, hence again the theorem holds. We may therefore assume that $n \geqslant 4\alpha - 2$.

By Theorem 1.2 we may also assume that

$$\frac{n + \omega(G)}{2\alpha - 1} < \frac{n(1 + c)}{2\alpha - 1},$$

i.e. $\omega(G) < nc$.

Case 1: *G is disconnected*: Let G be divided into two non-empty parts G_1 and G_2 with no edges between them. Let $\alpha(G_i) = \alpha_i$, $c_i = \frac{1}{4\alpha_i-3}$ and $|V(G_i)| = n_i$ for $i = 1, 2$. Then $n = n_1 + n_2$ and $\alpha = \alpha_1 + \alpha_2$, and $c_i > c$.

If $\alpha_1 \geqslant 3$ and $\alpha_2 \geqslant 3$ then by induction we are through unless

$$\frac{n_i(1 + c)}{2\alpha_i - 1} < \frac{n(1 + c)}{2\alpha - 1} \quad \text{for } i = 1, 2.$$

But in this case we have

$$n_i(2\alpha - 1) < n(2\alpha_i - 1) \quad \text{for } i = 1, 2.$$

By adding these we get

$$n(2\alpha - 1) < n(2\alpha - 2)$$

which is a contradiction.

If $\alpha_1 = 2$ and $\alpha_2 = \alpha - 2 \geqslant 3$, then by the theorem of Duchet and Meyniel and induction we are through unless both

$$\frac{n_1}{3} < \frac{n(1 + c)}{2\alpha - 1}$$

and

$$\frac{n_2(1 + c)}{2\alpha - 5} < \frac{n(1 + c)}{2\alpha - 1}.$$

Hence

$$n_1 < \frac{n(3+3c)}{2\alpha - 1}$$

and

$$n_2 < \frac{n(2\alpha - 5)}{2\alpha - 1}.$$

Adding these and multiplying by $2\alpha - 1$ we get

$$n(2\alpha - 1) < n(3 + 3c + 2\alpha - 5)$$

hence

$$1 < 3c,$$

which is a contradiction.

If $\alpha_1 = 1$ and $\alpha_2 = \alpha - 1 \geqslant 3$, then we are through unless

$$n_1 < \frac{n(1+c)}{2\alpha - 1}$$

and

$$\frac{n_2(1+c)}{2\alpha - 3} < \frac{n(1+c)}{2\alpha - 1}.$$

Working as before we get

$$n(2\alpha - 1) < n(1 + c + 2\alpha - 3)$$

and

$$n(2\alpha - 1) < n(2\alpha - 2 + c),$$

which again gives a contradiction.

If $\alpha_1 = 2$ and $\alpha_2 = 2$ then $\alpha = 4$ and we are through unless

$$\frac{n_i}{3} < \frac{n(1+c)}{7} \quad \text{for } i = 1, 2.$$

Adding these two inequalities gives

$$7n < n(6 + 6c),$$
$$n < n \cdot 6c,$$

which is a contradiction since $c = \frac{1}{4\alpha - 3} = \frac{1}{13}$.

Finally, if $\alpha_1 = 1$ and $\alpha_2 = 2$, then $\alpha = 3$ and we are through unless

$$n_1 < \frac{n(1+c)}{5}$$

and

$$\frac{n_2}{3} < \frac{n(1+c)}{5}.$$

This gives

$$5n < n(4+4c),$$
$$n < n \cdot 4c,$$

which is a contradiction since $c = \frac{1}{4\alpha-3} = \frac{1}{9}$.

This finishes the proof in Case 1.

Case 2: *G is connected and contains an induced claw C* (i.e. a $K_{1,3}$):We build a set $T \subseteq V(G)$ starting from the claw C by the method of Duchet and Meyniel as we did in the proof of Theorem 1.2. The obtained T satisfies

(i) $V(C) \subseteq T \subseteq V(G)$;
(ii) $|T| = 2\alpha' - 2$;
(iii) $\alpha(G[T]) = \alpha' \geqslant 3$;
(iv) T is dominating in G;
(v) $G[T]$ is connected.

(ii) is obtained using the claw C; the following argument needs that $|T| \leqslant 2\alpha - 2$. Contract T into a single vertex and contract $G - T$ into as large a complete graph K_{k-1} as possible. Thus we get a complete k-graph as minor in G. We shall estimate k.

Since $|V(G)| \geqslant 4\alpha - 2$ and $|T| \leqslant 2\alpha - 2$ we have that $|V(G-T)| \geqslant 2\alpha$.

For $\alpha(G-T) \geqslant 3$ we get by induction

$$\begin{aligned} k &\geqslant \left\lceil \frac{(n-2\alpha'+2)(1+c)}{2\alpha(G-T)-1} \right\rceil + 1 \\ &\geqslant \left\lceil \frac{(n-2\alpha'+2)(1+c)+2\alpha-1}{2\alpha-1} \right\rceil \\ &\geqslant \left\lceil \frac{n(1+c)}{2\alpha-1} + \frac{1-(2\alpha-2)c}{2\alpha-1} \right\rceil \\ &\geqslant \left\lceil \frac{n(1+c)}{2\alpha-1} \right\rceil. \end{aligned}$$

For $\alpha(G-T) = 2$ we get by the theorem of Duchet and Meyniel

$$\begin{aligned} k &\geqslant \left\lceil \frac{n-2\alpha'+2}{3} \right\rceil + 1 \\ &\geqslant \left\lceil \frac{n-2\alpha+5}{3} \right\rceil \\ &\geqslant \left\lceil \frac{n(1+c)}{2\alpha-1} \right\rceil. \end{aligned}$$

The last inequality holds if

$$n(2\alpha - 4 - 3c) > (2\alpha-1)(2\alpha-5)$$

which does hold because $n \geqslant 4\alpha - 2$ (from the start of the proof) and $c = \frac{1}{4\alpha-3} \leqslant \frac{1}{9}$.

For $\alpha(G-T)=1$ we get

$$k \geqslant n-2\alpha'+2+1 \geqslant n-2\alpha+3 \geqslant \left\lceil \frac{n(1+c)}{2\alpha-1} \right\rceil.$$

This proves the theorem in Case 2.

Case 3: *G is connected and claw-free*: Let x denote any vertex of G and let P and Q denote the neighbours and non-neighbours of x, respectively. Put $p=|P|$ and $q=|Q|$, so that $n=p+q+1$.

The graph $G[P]$ has $\alpha(G[P]) \leqslant 2$ since G is claw-free. Therefore, by Theorem 1.2, G has a complete k-graph as minor, where

$$k \geqslant \frac{p+\omega(G[P])}{3}+1.$$

If $\frac{p+\omega(G[P])}{3}+1 \geqslant \frac{n(1+c)}{2\alpha-1}$ we are through, hence we may assume

$$\frac{p+\omega(G[P])}{3}+1 < \frac{n(1+c)}{2\alpha-1},$$

i.e.

$$(2\alpha-1)p < 3n(1+c)-(2\alpha-1)(3+\omega(G[P])).$$

In $G[P]$ consider a set Z of as many pairwise disjoint-induced paths of length 2 as possible. The remaining part of $G[P]$ consists of say d vertices. It is not difficult to se that it induces one or two complete graphs, hence $0 \leqslant d \leqslant 2\omega < 2nc$ (remember that we proved that $\omega < nc$ just before starting on Case 1). Contract in G each of the induced 2-paths in Z to a single vertex. Remove the vertex x and the set of d vertices in P from the obtained graph to get the graph G'. The vertices obtained by the contractions form a complete graph, since $\alpha(G[P]) \leqslant 2$. Moreover $\alpha(G') \leqslant \alpha-1$, since α independent vertices in G' would give $\alpha+1$ independent vertices in G. By selecting x as a member of a maximum independent set in G we may in fact assume $\alpha(G')=\alpha-1 \geqslant 2$.

We shall now use Theorem 1.2 on the graph G' to get a complete k-graph minor of G', and thus of G, where

$$\begin{aligned} k &\geqslant \frac{|V(G')|+\omega(G')}{2\alpha(G')-1} \\ &\geqslant \frac{\frac{p-d}{3}+q+\frac{p-d}{3}}{2\alpha-3} \\ &= \frac{2p-2d+3q}{3(2\alpha-3)}. \end{aligned}$$

If $\frac{2p-2d+3q}{3(2\alpha-3)} \geqslant \frac{n(1+c)}{2\alpha-1}$, we are through, hence we may assume

$$\frac{2p-2d+3q}{3(2\alpha-3)} < \frac{n(1+c)}{2\alpha-1}.$$

From this we get

$$\frac{3n-3-p-2d}{3(2\alpha-3)} < \frac{n(1+c)}{2\alpha-1},$$

i.e.

$$(3n-3-p-2d)(2\alpha-1) < 3n(1+c)(2\alpha-3).$$

A simple calculation gives

$$6n-(3+2d)(2\alpha-1)-3nc(2\alpha-3) < p(2\alpha-1).$$

Earlier we proved that

$$p(2\alpha-1) < 3n(1+c)-(2\alpha-1)(3+\omega(G[P])).$$

Combining the last two inequalities we get

$$3n < 3nc(2\alpha-2)+(2d-\omega(G[P]))(2\alpha-1).$$

Since $d \leqslant 2nc$ and $\omega(G[P]) \geqslant \frac{d}{2}$ we get

$$3n < 3nc(2\alpha-2+2\alpha-1),$$
$$\frac{1}{4\alpha-3} < c.$$

This contradicts our assumption that $c = \frac{1}{4\alpha-3}$.

This final contradiction proves Theorems 1.3 and 1.4. $\square$

The obtained $c = \frac{1}{4\alpha-3}$ seems to be the best possible constant that may be obtained by the method used here.

4. Proof of Theorem 1.5

Let G be a graph on n vertices with $\alpha(G) \leqslant 2$ and containing a connected matching M of size $\lceil kn \rceil$, where k is a positive constant.

In the graph $G - V(M)$ let z denote the maximum number of pairwise disjoint induced paths of length 2. The remaining part of G consists of one or two complete graphs, say on d vertices altogether. Then

$$n = 2 \cdot |M| + 3z + d = 2\lceil kn \rceil + 3z + d.$$

Moreover it is easy to see that G has $K_{\lceil kn \rceil + z}$ as a minor, and that G has $K_{\lceil \frac{n+w}{3} \rceil} \supseteq \mathrm{K}_{\lceil \frac{n}{3}+\frac{d}{6} \rceil}$ as a minor, where the second statement follows from Theorem 1.2.

If $d \geqslant \frac{2}{3}kn$, then

$$\frac{n}{3}+\frac{d}{6} \geqslant \frac{n}{3}+\frac{kn}{9}$$

and if $d \leqslant \frac{2}{3}\,kn$, then

$$\begin{aligned}\lceil kn\rceil + z &= \lceil kn\rceil + \frac{n-2\lceil kn\rceil - d}{3}\\ &= \frac{1}{3}\lceil kn\rceil + \frac{n}{3} - \frac{d}{3}\\ &\geqq \frac{1}{3}\lceil kn\rceil + \frac{n}{3} - \frac{2}{9}\lceil kn\rceil\\ &\geqslant \frac{n}{3} + \frac{1}{9}\lceil kn\rceil\\ &\geqslant \frac{n}{3} + \frac{kn}{9}.\end{aligned}$$

This proves the first part of Theorem 1.5.

As for the second part, let G be a graph on n vertices with $\alpha(G) \leqslant 2$ having $K_{\lceil cn\rceil}$ as a minor for some constant $c > \frac{1}{3}$. Then at least a linear fraction of the vertices of $K_{\lceil cn\rceil}$ must be obtained from sets of vertices of G of size at most 2. More formally:

Let us suppose that the minor $K_{\lceil cn\rceil}$ is obtained by contracting x connected subgraphs each of size at least 3 into separate single vertices, and contracting y disjoint edges into y vertices and keeping z single vertices, where

$$\lceil cn\rceil = x + y + z$$

and

$$n \geqslant 3x + 2y + z.$$

Clearly, the graph G has a connected matching of size at least $y + \left\lfloor \frac{z}{2}\right\rfloor$.

If $z \geqslant \left(\frac{3c-1}{2}\right) n$ then

$$y + \left\lfloor \frac{z}{2}\right\rfloor \geqslant \frac{z-1}{2} \geqslant \left(\frac{3c-1}{4}\right) n - \frac{1}{2}.$$

If $z \leqslant \left(\frac{3c-1}{2}\right) n$ then

$$\begin{aligned}3cn &\leqslant 3\lceil cn\rceil\\ &= 3x + 3y + 3z\\ &= (3x + 2y + z) + y + 2z\\ &\leqslant n + y + 2z.\end{aligned}$$

Hence

$$(3c-1)n \leqslant y + 2z$$

and therefore

$$y + \left\lfloor \frac{z}{2}\right\rfloor \geqslant (3c-1)n - \frac{3}{2}z - \frac{1}{2}$$

$$\geqslant (3c-1)n - \frac{3}{2}\left(\frac{3c-1}{2}\right)n - \frac{1}{2}$$
$$= \left(\frac{3c-1}{4}\right)n - \frac{1}{2}.$$

This proves Theorem 1.5. □

5. Remarks on the case $\alpha = 2$

Recall that a matching M in a graph G is called *connected* if any two matching edges are joined by at least one edge in G. It is called *dominating* if every vertex of $G - V(M)$ is adjacent to at least one endvertex of each edge of M. It is easy to see that a smallest counterexample to Hadwiger's conjecture for $\alpha = 2$ does not contain a non-empty connected dominating matching (statement (12) in [7]). This motivates the search for connected dominating matchings in graphs G with $\alpha(G) = 2$ (they do not always exist—for example take G equal to the union of two disjoint complete graphs).

Lemma 5.1. *Let G be a graph with $\alpha(G) = 2$ and let S be a minimum separating set of G. Then $G - S$ consists of two complete subgraphs with vertex sets A and B. Moreover, for any subset S' of S with $|S'| \leqslant |A|$, there exists a matching M in G of size $|S'|$ consisting of edges from A to S'* (*i.e. a complete matching of S' into A*).

Proof of Lemma 5.1. The first half of the conclusion is obvious. Suppose the second is not true. Then by Hall's Theorem there is a subset D of S with neighbour set N in A satisfying $|N| < |D| \leqslant |A|$. Then $(S - D) \cup N$ is a smaller separating set in G than S. This contradiction proves Lemma 5.1. □

Each of the vertices of S is joined completely to either A or B or both. Let S_A be the vertices joined completely to A and let $S'_A = S - S_A$. Define S_B and S'_B similarly, and note that $S'_A \subseteq S_B$ and $S'_B \subseteq S_A$.

The pair A and S'_A satisfies the last sentence of Lemma 5.1. Since each vertex of S'_A has a non-neighbour in A, and since by Lemma 5.1 each vertex of A has at most $|A| - 1$ non-neighbours in S'_A, it follows that

$$|S'_A| \leqslant |A|(|A| - 1).$$

If there is a non-empty matching M of edges from A to S'_A that dominates all vertices of $S'_A - V(M)$, then M is a non-empty, connected and dominating matching of the whole graph G, as is easily seen. This is of course the case if $1 \leqslant |S'_A| \leqslant |A|$. Hence, the interesting cases to consider are those where $|S'_A| = 0$ or those where $|A| < |S'_A| \leqslant |A|(|A| - 1)$, and similarly for B.

If $|S'_A| = |S'_B| = 0$, then let M consist of any one edge from A to S. Then M is a non-empty connected dominating matching.

We may therefore assume $|S'_A| \neq 0$. If $|S'_B| = 0$ and $S'_A \neq S$, then any edge from B to $S - S'_A$ is a non-empty connected dominating matching, hence if $|S'_B| = 0$ we may assume $S'_A = S$.

Hence the interesting cases to consider are

(i) $|A| < |S'_A| \leqslant |A|(|A|-1)$, $|S'_B| = 0$ and $S'_A = S$,
(ii) $|A| < |S'_A| \leqslant |A|(|A|-1)$ and $|B| < |S'_B| \leqslant |B|(|B|-1)$.

The above discussion motivates Theorem 5.2, in which G and S are used to denote what in the above terminology would be $G[A \cup S'_A]$ and S'_A.

Theorem 5.2. *Suppose G is a connected graph with $\alpha(G) = 2$. Suppose further that $V(G) = A \cup S$ such that*

(i) *A and S are disjoint and both non-empty,*
(ii) $|A| \leqslant 3$,
(iii) *$G[A]$ is complete*, and
(iv) *for all $S' \subseteq S$ with $1 \leqslant |S'| \leqslant |A|$ there exists a complete matching of S' into A.*

Then there exists a non-empty matching M of edges from A to S, dominating all vertices of G outside the matching.

Unfortunately Theorem 5.2 is not true for $|A| = 4$. A counterexample may be obtained as follows:

Let $A = \{a_1, a_2, a_3, a_4\}$ and let S consist of 12 vertices b_{i1}, b_{i2} and b_{i3}, $i = 1, 2, 3, 4$. Let $G[A] = K_4$, the vertex a_i be joined to all vertices of S except b_{i1}, b_{i2} and b_{i3}. Moreover let b_{i1} and b_{i2}, $i = 1, 2, 3, 4$, span a K_8, let b_{i3}, $i = 1, 2, 3, 4$, span a K_4, and let b_{i3} be joined to b_{j1} and b_{j2} only when $i = j$. This example satisfies the conditions of Theorem 5.2, but not the conclusion. This can be seen by a straightforward case analysis.

The above example is not a smallest possible counterexample to Theorem 5.2 with the condition $|A| \leqslant 3$ removed. Another counterexample (the smallest?) with $|A| = 4$ and $|S| = 9$ exists. Let again $A = \{a_1, a_2, a_3, a_4\}$ and let $S = \{s_{12}, s_{13}, s_{23}, s_{124}, s'_{124}, s_{134}, s'_{134}, s_{234}, s'_{234}\}$. For $i = 1, 2, 3, 4$ a_i is joined to exactly those s and s' with an index i. Moreover, $G[A] = K_4$ and $G[S]$ contains all possible 36 edges except the 12 edges $s_{ij}s_{hk4}$ and $s_{ij}s'_{hk4}$, where $\{i, j\} \neq \{h, k\}$.

Let us finally prove Theorems 5.2 and 1.6.

Proof of Theorem 5.2. For $|A| \geqq |S|$ a complete matching of S into A will suffice. Hence suppose $|A| < |S|$.

For $|A| = 1$ the condition (iv) implies that A is completely joined to S, and hence any one edge from A to S constitutes a matching M of the type desired.

For $|A| = 2$ we let $A = \{a_1, a_2\}$. Moreover, let S_i denote the vertices of S joined to only a_i in A, $i = 1, 2$, and let S_{12} be the vertices of S joined to both a_1 and a_2. Then by (iv) $|S_i| \leqslant 1$. Let M consist of an edge joining a_1 to S_1, if it exists, together with an edge joining a_2 to S_2, if it exists. Then if $M \neq \emptyset$, M satisfies the theorem. Otherwise, let M consist of a single edge from a_1 to S_{12}, and again the desired conclusion is obtained.

Finally, suppose $|A| = 3 < |S|$. Let $A = \{a_1, a_2, a_3\}$. For $i = 1, 2, 3$ let $S_i = \{s \in S \mid s$ is joined to a_i, and to only a_i, in $A\}$, $S_{ij} = \{s \in S \mid s$ is joined to a_i and to a_j, and to only these two, in $A\}$ and $S_{123} = \{s \in S \mid s$ is completely joined to $A\}$.

By (iv) $|S_i| \leqslant 1$ for $i = 1, 2, 3$, and $|S_{ij}| \leqslant 2$ for all $1 \leqslant i < j \leqslant 3$. We also know from (iv) that

$$|S_i| + |S_j| + |S_{ij}| \leqslant 2, \quad 1 \leqslant i < j \leqslant 3. \tag{5.1}$$

Adding these three inequalities we get

$$2|S_1| + 2|S_2| + 2|S_3| + |S_{12}| + |S_{13}| + |S_{23}| \leqslant 6. \tag{5.2}$$

If $S = S_{123}$ then any edge joining A to S is a dominating matching of the type we seek. If $1 \leqslant |S - S_{123}| \leqslant 3$ then a complete matching of $S - S_{123}$ into A is a dominating matching of the type we seek. Now $|S| = |S_1| + |S_2| + |S_3| + |S_{12}| + |S_{13}| + |S_{23}| + |S_{123}|$, so we may assume that

$$|S - S_{123}| = |S_1| + |S_2| + |S_3| + |S_{12}| + |S_{13}| + |S_{23}| \geqslant 4. \tag{5.3}$$

It follows from (5.2) and (5.3) that

$$|S_1| + |S_2| + |S_3| \leqslant 2. \tag{5.4}$$

We propose to treat separate cases.

Case 1: $|S_1| + |S_2| + |S_3| = 2$: Since $|S_i| \leqslant 1$ we may assume that $|S_1| = |S_2| = 1$ and $|S_3| = 0$. By (5.1) it follows that $|S_{12}| = 0$ and by (5.1) and (5.3) that $|S_{13}| = |S_{23}| = 1$. Then $M = \{a_1s_1, a_3s_{23}\}$ suffices, since s_2 is joined to s_1 and s_{23} (as $\alpha = 2$) and s_{13} is joined to a_1 and a_3. (We use here the convention that $S_i = \{s_i\}$ and $S_{ij} = \{s_{ij}\}$ when $|S_i| = 1$ and $|S_{ij}| = 1$. If $|S_{ij}| = 2$ we shall use $S_{ij} = \{s_{ij}, s'_{ij}\}$.)

Case 2: $|S_1| + |S_2| + |S_3| = 1$: We may suppose that $|S_1| = 1$ and $|S_2| = |S_3| = 0$. By (5.1), $|S_{12}| \leqslant 1$, $|S_{13}| \leqslant 1$ and $|S_{23}| \leqslant 2$. By (5.3), $|S_{12}| + |S_{13}| + |S_{23}| \geqslant 3$. Hence we may assume that $|S_{12}| = 1$ and $1 \leqslant |S_{23}| \leqslant 2$.

If $|S_{13}| = 0$ then $M = \{a_2s_{12}\}$ suffices. Hence we may assume that $|S_{13}| = 1$. But then $M = \{a_2s_{12}, a_3s_{13}\}$ suffices.

Case 3: $|S_1| = |S_2| = |S_3| = 0$: In this case $|S_{ij}| \leqslant 2$ and $4 \leqslant |S_{12}| + |S_{13}| + |S_{23}|$ by (5.1) and (5.3). We may assume that $|S_{12}| = 2$. If $|S_{23}| = 0$, then $M = \{a_1s_{12}\}$ suffices. Likewise if $|S_{13}| = 0$. Hence $1 \leqslant |S_{13}| \leqslant 2$ and $1 \leqslant |S_{23}| \leqslant 2$.

If $|S_{13}| = |S_{23}| = 1$, then $M = \{a_1s_{13}, a_2s_{23}\}$ suffices.

If $|S_{23}| = 2$ and $|S_{13}| = 1$, then $M = \{a_2s_{12}, a_3s_{13}\}$ suffices if $s_{13}s'_{12}$ is an edge of G and $S_{12} = \{s_{12}, s'_{12}\}$. Hence we may assume that $s_{13}s_{12}$ is not an edge in G. By symmetry we may assume that s_{13} is not joined to any vertex of $S_{12} \cup S_{23}$. Since $\alpha = 2$, it follows that S_{12} and S_{23} are completely joined, and hence $M = \{a_1s_{12}, a_3s_{23}\}$ suffices.

Finally, let $|S_{12}| = |S_{23}| = |S_{13}| = 2$. Since $\alpha = 2$, each S_{ij} induces a K_2.

If s_{12} is joined completely to S_{13} then $M = \{a_2s_{12}\}$ suffices. Hence we may assume by symmetry that any two of S_{12}, S_{13} and S_{23} are joined by either no edge at all, exactly one edge, or exactly two disjoint edges. But if two of S_{12}, S_{23} and S_{13} are joined by no edge at all or exactly one edge, then S has 3 independent vertices contrary to $\alpha = 2$.

Hence any two of S_{12}, S_{23} and S_{13} are joined by exactly two disjoint edges. Since $\alpha = 2$ this implies that S induces a prism in G consisting of two triangles joined by a three matching edges.

We may assume that s_{12}, s_{13}, s_{23} span one triangle and $s'_{12}, s'_{13}, s'_{23}$ the other. Then $M = \{a_1 s_{12}, a_3 s'_{23}\}$ suffices.

This proves Theorem 5.2. $\square$

Proof of Theorem 1.6. Let G be a graph on n vertices with $\alpha(G) = 2$. Suppose G is $(n-4)$-connected. Then any induced subgraph G' of G on say n' vertices, is either complete or $(n' - 4)$-connected. Let S be a minimum separating set of G. With the notation introduced in this section we have $|A| + |B| \leqslant 4$, and hence $|A| \leqslant 3$ and $|B| \leqslant 3$.

If $S'_A \neq \emptyset$, then by Theorem 5.2 used on A and S'_A we obtain a non-empty connected dominating matching M in $G[A \cup S'_A]$, and then M is dominating in the whole of G since $S'_A \subseteq S_B$. Since $G' = G - V(M)$ is $(n - 2|M| - 4)$-connected, it has a $K_{\lceil \frac{n}{2} \rceil - |M|}$-minor by induction, and hence G has a $K_{\lceil \frac{n}{2} \rceil}$-minor.

If $S'_B \neq \emptyset$, we have likewise a $K_{\lceil \frac{n}{2} \rceil}$-minor of G.

If $S'_A = \emptyset$ and $S'_B = \emptyset$, then any edge from A to S is a non-empty connected dominating matching, and hence we are through again.

This proves Theorem 1.6. $\square$

6. Conclusion

For $\alpha = 2$ no improvement of the constant $\frac{1}{3}$ in the theorem of Duchet and Meyniel has been achieved. This is an interesting open problem due to Paul Seymour. For all $\alpha \geqslant 3$ the present paper gives a slight improvement of the constant $\frac{1}{2\alpha-1}$ of Duchet and Meyniel. It would be interesting if further improvements could be obtained.

The interest of the special case $\alpha = 2$ of Hadwiger's conjecture was first brought to our attention by Wolfgang Mader (oral communication). We are also grateful to Michael Stiebitz for stimulating discussions and for suggesting to us the possibility of a theorem along the lines of Theorem 5.2.

We also wish to thank two very helpful referees for numerous suggestions that has significantly improved our exposition. It was also pointed out that in fact the theorem of Duchet and Meyniel may be slightly improved in a different way:

Revised Theorem of Duchet and Meyniel. *Any non-empty graph G on n vertices and at least one edge has $K_{\lceil (n+2\alpha(G)-2)/(2\alpha(G)-1) \rceil}$ as a minor.*

It seems likely that some of our results, e.g. Theorems 1.2 and 1.4, may have similar improvements (this does not improve the constant factor on n however; and this improvement was our main goal here). However, such improvements might perhaps make some of our computations more straightforward.

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